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Characterization of Weights in Best Rational Weighted Approximation of Piecewise Smooth Functions, II

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1. INTRODUCTION

The problem of characterization of weights in weighted L_p rational approximation of piecewise smooth functions f was introduced and studied in [4]. A motivation for the study of this subject is its relationship to the realization of recursive filters. In practice, it is sometimes desirable to include a multiplicative factor s with the rational approximant r_n . This leads to the so-called generalized inverse approximation problem (cf. [1,3]). An example is $f \equiv 1$, and in this case r_n provides an inverse approximation of $1/s$, which is a generalization of the least-squares inverse approximation that guarantees stability [2]. To facilitate our discussion, we need some notation and definitions.

Let

$$I: 0 = x_0 < x_1 < \cdots < x_{m+1} = 1$$

be a partition of the interval $[0,1]$. As in [4], we will also use I to denote

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the set $\{x_1, \dots, x_m\}$ of interior partition points. Let $U = \{u_1, \dots, u_m\}$ be a system of non-negative integers and denote by $A(\Gamma, U)$ the collection of all complex-valued continuous functions f on $[0, 1]$ whose restrictions on each $I_j = [x_j, x_{j+1}]$ are analytic on I_j , $j = 1, \dots, m$, and satisfy the joining conditions

$$f^{(s)}(x_j^-) = f^{(s)}(x_j^+), \quad s = 0, \dots, u_j,$$

and

$$f^{(u_j+1)}(x_j^-) \neq f^{(u_j+1)}(x_j^+).$$

If $U = 0 := \{0, \dots, 0\}$, then we will simply write $A(\Gamma, U) = A(\Gamma)$. Let w denote an arbitrary weight function, i.e., w is measurable and $0 < w < \infty$ a.e. on $[0, 1]$. For any measurable function f defined on $[0, 1]$, we will use the notation

$$\|f\|_{L_p(w)} = \begin{cases} \left\{ \int_0^1 |f(x)|^p w(x) dx \right\}^{1/p} & \text{if } 0 < p < \infty, \\ \text{ess sup}_{0 \leq x \leq 1} |f(x)| w(x) & \text{if } p = \infty. \end{cases}$$

Of course, if $1 \leq p \leq \infty$, $\|\cdot\|_{L_p(w)}$ defines a norm for the space $L_p(w)$ of functions f with $\|f\|_{L_p(w)} < \infty$. Let $\mathbf{R}_n[a, b]$ denote the collection of all rational functions p_n/q_n where p_n and q_n are in π_n , the set of all polynomials of degree $\leq n$, and are relatively prime, with $q_n(x) \neq 0$ for all x in $[a, b]$. In addition, set $\mathbf{R}_n = \mathbf{R}_n[0, 1]$ and $\mathbf{R} = \bigcup_n \mathbf{R}_n$. Let

$$\Delta: 0 = y_0 < y_1 < \dots < y_{l+1} = 1$$

be another partition of $[0, 1]$ and $V = \{v_1, \dots, v_l\}$ the corresponding system of non-negative integers. Let s be a fixed function in the class $A(\Delta, V)$. The "distance" of f from $s\mathbf{R}_n$ will be denoted by

$$e_n(s, f)_{L_p(w)} := \inf \{ \|f - sr_n\|_{L_p(w)} : r_n \in \mathbf{R}_n \},$$

where $0 < p \leq \infty$. We also need the following notation introduced in [4]. For any weight function w on $[0, 1]$, set

$$U_p(w) = \left\{ x \in [0, 1] : \int_{[x-\delta, x+\delta] \cap [0, 1]} w(t) dt = \infty, \text{ for all } \delta > 0 \right\}$$

if $0 < p < \infty$ and

$$U_\infty(w) = \left\{ x \in [0, 1] : \text{ess sup}_{[x-\delta, x+\delta] \cap [0, 1]} w(x) = \infty, \text{ for all } \delta > 0 \right\}$$

For any systems $\Theta = \{\theta_1, \dots, \theta_k\}$ and $\mathcal{M} = \{\mu_1, \dots, \mu_k\}$ with $0 \leq \theta_1 < \dots \leq \theta_k \leq 1$ and $\mu_1, \dots, \mu_k > 0$, denote by $W_p(\Theta, \mathcal{M})$, $0 < p \leq \infty$, the collection of all weight functions w on $[0, 1]$ that satisfy the conditions

$$U_p(w) = \Theta$$

and

$$\prod_{s=1}^k |\cdot - \theta_s|^{\mu_s} \in L_p(w).$$

The main result in this paper can be stated as follows.

THEOREM 1. *Let the classes $A(\Gamma, U)$ and $A(\Delta, V)$ be defined as above, s a fixed function in $A(\Delta, V)$, $0 < p \leq \infty$, and w a given weight function on $[0, 1]$. Then a necessary and sufficient condition for $e_n(s, f)_{L_p(w)} \rightarrow 0$ as $n \rightarrow \infty$ where f is an arbitrary function in $A(\Gamma, U) \setminus \mathbf{R}$, is that there exist Θ and \mathcal{M} such that $w \in W_p(\Theta, \mathcal{M})$ and the following conditions are satisfied:*

(i) *The set $\Phi = \{x \in [0, 1] : s(x) = 0\}$ is finite and $\Phi \cap U_p(w) = \emptyset$; furthermore, if $p = \infty$, then for every $\varphi \in \Phi$*

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,sup}_{x \in [\varphi - \delta, \varphi + \delta] \cap [0, 1]} w(x) = 0.$$

(ii) *If $\theta_j = x_{s_1} \in \Gamma$, then*

$$\lim_{\delta \rightarrow 0^+} \|\chi_{[\theta_j - \delta, \theta_j]}(\cdot - \theta_j)^{\mu_{s_1} + 1}\|_{L_p(w)} = 0 \quad (1)$$

or

$$\lim_{\delta \rightarrow 0^+} \|\chi_{[\theta_j, \theta_j + \delta]}(\cdot - \theta_j)^{\mu_{s_1} + 1}\|_{L_p(w)} = 0. \quad (2)$$

(iii) *If $\theta_j = y_{s_2} \in \Delta$, then*

$$\lim_{\delta \rightarrow 0^+} \|\chi_{[\theta_j - \delta, \theta_j]}(\cdot - \theta_j)^{\nu_{s_2} + 1}\|_{L_p(w)} = 0$$

or

$$\lim_{\delta \rightarrow 0^+} \|\chi_{[\theta_j, \theta_j + \delta]}(\cdot - \theta_j)^{\nu_{s_2} + 1}\|_{L_p(w)} = 0.$$

Here and throughout, χ_J denotes, as usual, the characteristic function of the set J .

2. PROOF OF THE NECESSITY CONDITION

Suppose that

$$e_n(s, f)_{L_p(w)} \rightarrow 0 \quad (3)$$

as $n \rightarrow \infty$, for any $f \in A(\Gamma, U) \setminus \mathbf{R}$. The proof of the existence of Θ and \mathcal{M} such that $w \in W_p(\Theta, \mathcal{M})$ is similar to that given in [4] (of course, it is possible that both of Θ and \mathcal{M} are empty). Next we prove the necessity of the conditions in (i). Assume that Φ is an infinite set. Then s vanishes identically on some interval $I_j = [x_j, x_{j+1}]$. Let $f \in A(\Gamma, U) \setminus \mathbf{R}$ with $f(x) = 1$ for $x \in I_j$. By (3), there exists a sequence $\{r_n\} \subset \mathbf{R}$ such that

$$\|\chi_{I_j}\|_{L_p(w)} \leq \|f - s r_n\|_{L_p(w)} \rightarrow 0.$$

It follows that $w(x) = 0$ for almost all x on I_j , which is a contradiction to our assumption on w . Set $\Phi = \{\varphi_1, \dots, \varphi_q\}$ and let $\{r_n\}$ be a sequence in \mathbf{R} such that

$$\rho_n := \|f_0 - s r_n\|_{L_p(w)} \rightarrow 0,$$

where $f_0 \in A(\Gamma, U)$ and satisfies $f_0(x) > 1$ for all $x \in [0, 1]$. For every fixed n , there exists a positive δ_n such that

$$\max_j \sup_x |\chi_{[\varphi_n - \delta_n, \varphi_j + \delta_n] \cap [0, 1]}(x) s(x) r_n(x)| \leq 1/2.$$

Thus we obtain

$$\sum_{j=1}^q \|\chi_{[\varphi_i - \delta_n, \varphi_j + \delta_n] \cap [0, 1]}\|_{L_p(w)} \leq 2q\rho_n \rightarrow 0$$

as $n \rightarrow \infty$. It is easy to see that this is equivalent to the conditions in (i).

Now we will show that conditions (ii) and (iii) are also necessary.

Let us first assume that $\theta_j = x_{s_1} \in \Gamma \setminus \Delta$. Since $\Phi \cap U_p(w) = \emptyset$, there exists a small positive δ^* such that s does not vanish on $[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]$, and without loss of generality, we may assume that on this set $s > \varepsilon^* > 0$, and since $\theta_j \notin \Delta$, that s is analytic there. By [4], there exist $r_n \in \mathbf{R}$ such that

$$\rho_n^* := \left\| \left| \frac{f}{s} - r_n \right| \chi_{[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]} \right\|_{L_p(w)} \rightarrow 0, \quad (4)$$

where

$$f(x) = (x - \theta_j)_+^{u_j + 1}, \text{ for } x \in [\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1].$$

If both (1) and (2) do not hold, then r_n must be of the form

$$r_n(x) = \frac{(x - \theta_j)_+^{u_{s_1} + 1} p_{n-u_{s_1}-1}(x)}{q_n(x)},$$

where $p_{n-u_{s_1}-1} \in \pi_{n-(u_{s_1}+1)}$ and $q_n \in \pi_n$. Hence, by (4)

$$\begin{aligned} \rho_n^* &= \left\| \left| \frac{\operatorname{sgn}(\cdot - \theta_j) + 1}{2s(\cdot)} - \frac{p_{n-u_{s_1}-1}(\cdot)}{q_n(\cdot)} \right| \right. \\ &\quad \left. \times (\cdot - \theta_j)^{u_{s_1} + 1} \chi_{[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]}(\cdot) \right\|_{L_p(w)} \rightarrow 0 \end{aligned}$$

and it follows that

$$\left. \frac{d}{dx} \frac{p_{n-u_{s_1}-1}(x)}{q_n(x)} \right|_{x=\theta_j} = \infty$$

(cf. the proof of Theorem 1 in [4]). But this is impossible. Similarly, if $\theta_j = y_{s_2} \in \mathcal{A} \setminus \Gamma$, then we arrive at a similar contradiction when we assume that both (1) and (2) do not hold.

Now suppose that $\theta_j = x_{s_1} = y_{s_2} \in \Gamma \cap \mathcal{A}$, and set $v = \min(u_{s_1}, v_{s_2})$. Assume that both

$$\lim_{\delta \rightarrow 0^+} \|\chi_{[\theta_j - \delta, \theta_j]}(\cdot) (\cdot - \theta_j)^{v+1}\|_{L_p(w)} = 0 \quad (5)$$

and

$$\lim_{\delta \rightarrow 0^+} \|\chi_{[\theta_j, \theta_j + \delta]}(\cdot) (\cdot - \theta_j)^{v+1}\|_{L_p(w)} = 0 \quad (6)$$

do not hold. Then since $\Phi \cap U_p(w) = \phi$, we may assume that $s(x) > \varepsilon^* > 0$ on some small interval $[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]$. From (3), we obtain (4) for some $\{r_n\} \subset \mathbf{R}$. Set

$$f^*(x) = \frac{f(x)}{s(x)}, \quad x \in [\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1].$$

Then both of the restrictions of f^* on $[\theta_j - \delta^*, \theta_j]$ and $[\theta_j, \theta_j + \delta^*]$ are analytic on the corresponding intervals. Furthermore, $(d^s/dx^s) f^*(x)$, $s = 0, \dots, v$, are continuous at $x = \theta_j$ and

$$\frac{d^{v+1}}{dx^{v+1}} f^*(\theta_j^-) \neq \frac{d^{v+1}}{dx^{v+1}} f^*(\theta_j^+).$$

Set

$$p^*(x) = \sum_{s=0}^v \frac{1}{s!} \left(\frac{d^s}{dx^s} f^*(\theta_j) \right) (x - \theta_j)^s.$$

Then $f^* - p^*$ is of the form

$$f^*(x) - p^*(x) = g^*(x)(x - \theta_j)^{v+1} \quad \text{where } x \in [\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]$$

and g^* satisfies the inequality

$$g^*(\theta_j^-) \neq g^*(\theta_j^+).$$

If both (5) and (6) do not hold, then r_n must be of the form

$$r_n(x) = p^*(x) + \frac{(x - \theta_j)^{v+1} p_{n-(v+1)}(x)}{q_n(x)}.$$

From (4) it follows that

$$\left\| \left| g^*(\cdot) - \frac{p_{n-(v+1)}(\cdot)}{q_n(\cdot)} \right| (\cdot - \theta_j)^{v+1} \chi_{[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]}(\cdot) \right\|_{L_p(w)} \rightarrow 0,$$

yielding

$$\left. \frac{d}{dx} \frac{p_{n-(v+1)}(x)}{q_n(x)} \right|_{x=\theta_j} = \infty,$$

which is again a contradiction.

3. PROOF OF THE SUFFICIENCY CONDITION

In order to prove that the conditions in Theorem 1 are sufficient we need several lemmas. The first one was established in [4].

LEMMA 1. Let $\eta = \exp(-1/\sqrt{n})$, $\xi_1, \dots, \xi_q \in [-1, 0) \cup (0, 1]$, $\mu > 0$, and $\mu_j > 0$, $j = 1, \dots, q$. Then for any constants δ , B , C , ε , and $\varepsilon_1, \dots, \varepsilon_q$ satisfying

$$0 < \delta < 1/2, \quad 1 < B^{[\mu]+1} < e, \quad c > 1, \quad \text{and } \varepsilon > 0,$$

there exist rational functions $r_n \in R_{m_n}[-1, 1]$ with $m_n = n + O(\sqrt{n})$ such that

$$\begin{aligned}
& |\operatorname{sgn} x - r_n(x)| \\
&= \begin{cases} O(1) & \text{for } x \in [-\eta^n, \eta^n], \\ O\left(\left(\frac{B^{[\mu]+1}}{e}\right)^{\sqrt{n}}\right) \prod_{\xi_j > 0} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} |x - \varepsilon B^{-\sqrt{n}}|^{\mu} & \text{for } x \in [\eta^n, 1], \\ O\left(\left(\frac{B^{[\mu]+1}}{e}\right)^{\sqrt{n}}\right) \prod_{\xi_j < 0} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} |x - \varepsilon B^{-\sqrt{n}}|^{\mu} & \text{for } x \in [-1, -\eta^n], \\ O(C^{-\sqrt{n}}) \prod_{j=1}^q |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} & \text{for } \delta \leq |x| \leq 1, \end{cases}
\end{aligned}$$

where the “ O ” terms are independent of x .

The second lemma we need is a well known result of Bernstein.

LEMMA 2. Let f be analytic on $[a, b]$. Then there exists a sequence of polynomials p_n in π_n and a positive λ such that

$$\max_{a \leq x \leq b} |f(x) - p_n(x)| = O(e^{-\lambda n}).$$

LEMMA 3. Let $\delta, \beta > 0$ be given, $\Gamma = \{x_1, x_2\} \subset (0, 1)$, $U = \{u_1, u_2\}$ and $w \in W_p(\Theta, \mathcal{M})$, $0 < p \leq \infty$, for some $\Theta = \{\theta_1, \dots, \theta_k\}$ and $\mathcal{M} = \{\mu_1, \dots, \mu_k\}$. Suppose that f is a piecewise analytic function of the form

$$f(x) = (x - x_1)^{u_1+1} (x - x_2)^{u_2+1} g(x) \chi_{[x_1, x_2]}(x),$$

where g is analytic on the interval $I_1 = [x_1, x_2]$. Then there exists $r_n \in R_n$, $n \geq n_0$ such that

$$\|f - r_n\|_{L_p(w)} = O(A^{-\sqrt{n}} + \mathcal{E}_n(B)) \quad (7)$$

for some $A > 1$ and $B > 1$, where

$$\mathcal{E}_n(B) = \sum_{\theta_s = x_j \in \Gamma} \min(\mathcal{E}_{n,s}^-(B), \mathcal{E}_{n,s}^+(B))$$

with

$$\begin{aligned}
\mathcal{E}_{n,s}^-(B) &= (B^{-\sqrt{n}})^{u_j+1} \|\chi_{[\theta_s - \delta, \theta_s - B^{-\sqrt{n}}]}\|_{L_p(w)} \\
&\quad + \|\chi_{[\theta_s - B^{-\sqrt{n}}, \theta_s]}(\cdot)(\cdot - \theta_s)^{u_j+1}\|_{L_p(w)}
\end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{n,s}^+(B) = & (B^{-\sqrt{n}})^{\mu_j+1} \|\chi_{[\theta_s+B^{-\sqrt{n}}, \theta_s+\delta]}\|_{L_p(w)} \\ & + \|\chi_{[\theta_s, \theta_s+B^{-\sqrt{n}}]}(\cdot)(\cdot-\theta_s)^{\mu_j+1}\|_{L_p(w)}. \end{aligned}$$

Furthermore, $\{r_n\}$ converges uniformly to f on $[0, 1]$.

The proof of this lemma is similar to that of Lemma 4 in [4]. We assume, without loss of generality, that $\delta > 0$ is so small that g is analytic on $[x_1 - \delta, x_2 + \delta]$ and $\theta_s \notin [x_1 - \delta, x_1) \cup (x_2, x_2 + \delta]$, $s = 1, \dots, k$. Construct a polynomial p_0 of degree $\leq \sum_{\theta_s \in [x_1, x_2]}([\mu_s] + 1)$ such that

$$p_0(x) - g(x) = \prod_{\theta_s \in [x_1, x_2]} (x - \theta_s)^{[\mu_s] + 1} \tilde{g}(x),$$

where \tilde{g} is also analytic on $[x_1 - \delta, x_2 + \delta]$. By Lemma 2, there is a polynomial p_1 of degree $\leq K[\sqrt{n}] - \sum_s([\mu_s] + 1)$ such that

$$|\tilde{g}(x) - p_1(x)| = O(e^{-\sqrt{n}})$$

uniformly for $x \in [x_1 - \delta, x_2 + \delta]$. Set

$$p_2(x) = p_0(x) - \prod_{\theta_s \in [x_1, x_2]} (x - \theta_s)^{[\mu_s] + 1} p_1(x).$$

Then p_2 is a polynomial of degree $K[\sqrt{n}]$ and

$$p_2(x) - g(x) = O(e^{-\sqrt{n}}) \prod_{\theta_s \in [x_1, x_2]} |x - \theta_s|^{\mu_s} \quad (8)$$

uniformly for $x \in [x_1 - \delta, x_2 + \delta]$.

There are the following possible cases:

- (1) $\Theta \cap A = \emptyset$;
- (2) $x_1 = \theta_{s_0} \in \Theta$, $x_2 \notin \Theta$;
- (3) $x_1 \notin \Theta$, $x_2 = \theta_{s_0} \in \Theta$; or
- (4) both $x_1 = \theta_{s_1}$ and $x_2 = \theta_{s_2}$ belong to the set Θ .

For simplicity, we will only give the proof for the case (4) since the others can be verified similarly. Set

$$x' = x_1 - 2B^{-\sqrt{n}} \quad \text{and} \quad x'' = x_2 - 2B^{-\sqrt{n}},$$

where

$$B = \frac{1}{2} \min \left\{ 1 + \exp \left(\frac{1}{[\mu_{s_1}] + 1} \right), 1 + \exp \left(\frac{1}{[\mu_{s_2}] + 1} \right) \right\}.$$

If n is sufficiently large, then we have

$$\theta_s \notin [x', x_1) \cup [x'', x_2), \quad s = 1, \dots, k.$$

By (8), we see that

$$\begin{aligned} & \chi_{[x', x'']}(x)(x-x_1)^{u_1+1}(x-x_2)^{u_2+1}p_2(x) - f(x) \\ &= O(e^{-\sqrt{n}}) \prod_{s=1}^k |x-\theta_s|^{\mu_s} - \chi_{[x', x_1]}(x)(x-x_1)^{u_1+1}(x-x_2)^{u_2+1}p_2(x) \\ & \quad + \chi_{[x'', x_2]}(x)(x-x_1)^{u_1+1}(x-x_2)^{u_2+1}p_2(x). \end{aligned} \quad (9)$$

Write

$$\chi_{[x', x'']}(x) = \frac{1}{2} \{ \operatorname{sgn}(x-x') - \operatorname{sgn}(x-x'') \}.$$

By Lemma 3, there are rational functions \tilde{r} and \hat{r} of degree $n + O(\sqrt{n})$ such that

$$\begin{aligned} & |\tilde{r}(x) - \operatorname{sgn}(x-x')| \\ &= \begin{cases} O(1) & \text{for } x \in [0, 1], \\ O(B^{[\mu_{s_1}] + 1}/e)^{\sqrt{n}} \prod_{s=1}^k |x-\theta_s|^{\mu_s} & \text{for } |x-x'| \geq \eta^n \text{ and } x \in [0, 1], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^k |x-\theta_s|^{\mu_s} & \text{for } |x-x'| \geq \delta \text{ and } x \in [0, 1], \end{cases} \end{aligned}$$

and

$$\begin{aligned} & |\hat{r}(x) - \operatorname{sgn}(x-x'')| \\ &= \begin{cases} O(1) & \text{for } |x-x''| \leq \eta^n, \\ O(B^{[\mu_{s_2}] + 1}/e)^{\sqrt{n}} \prod_{s=1}^k |x-\theta_s|^{\mu_s} & \text{for } |x-x''| > \eta^n \text{ and } x \in [0, 1], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^k |x-\theta_s|^{\mu_s} & \text{for } |x-x''| \geq \delta \text{ and } x \in [0, -1], \end{cases} \end{aligned}$$

where C is an arbitrarily given positive constant. It is known that

$$p_2(x) = O(e^{\lambda' \sqrt{n}})$$

for some $\lambda' > 0$ uniformly for $x \in [0, 1]$. Set $C = \exp(\lambda' + 1)$ and

$$r^*(x) = \frac{1}{2} (\tilde{r}(x) - \hat{r}(x))(x-x_1)^{u_1+1}(x-x_2)^{u_2+1}p_2(x).$$

Then we obtain, using (9),

$$\begin{aligned} \|f - r^*\|_{L_p(w)} &= O(e^{-\sqrt{n}}) + O(B^{[\mu_{s_1}] + 1} e)^{\sqrt{n}} \\ &\quad + O(B^{[\mu_{s_2}] + 1} / e)^{\sqrt{n}} + O(\mathcal{E}_{n, s_1}^-(B)) + O(\mathcal{E}_{n, s_2}^-(B)), \end{aligned}$$

and this, in turn, assures the existence of $r_n \in \mathbf{R}_n$ such that

$$\|f - r_n\|_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathcal{E}_{n, s_1}^-(B) + \mathcal{E}_{n, s_2}^-(B))$$

for some $A > 1$ and $B > 1$. Similarly, there exist $r_n \in \mathbf{R}_n$ such that

$$\begin{aligned} \|f - r_n\|_{L_p(w)} &= O(A^{-\sqrt{n}}) + O(\mathcal{E}_{n, s_1}^-(B) + \mathcal{E}_{n, s_2}^+(B)), \\ \|f - r_n\|_{L_p(w)} &= O(A^{-\sqrt{n}}) + O(\mathcal{E}_{n, s_1}^+(B) + \mathcal{E}_{n, s_2}^-(B)), \end{aligned}$$

or

$$\|f - r_n\|_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathcal{E}_{n, s_1}^+(B) + \mathcal{E}_{n, s_2}^+(B)).$$

Hence, combining these estimates, we obtain (7). By the same proof, we can also conclude that $\{r_n\}$ converges uniformly to f on $[0, 1]$. This completes the proof of the lemma.

Remark 1. A similar proof also shows that the result in Lemma 3 also holds for $x_1 = 0$ and/or $x_2 = 1$.

We are now ready to prove that the conditions in Theorem 1 are sufficient.

Let $\Phi = \{\varphi_1, \dots, \varphi_q\}$ where $\varphi_1 < \dots < \varphi_q$. We will only consider the case where $\varphi_1 \neq 0$ and $\varphi_q \neq 1$, since the other cases can be verified in a similar manner. Choose small $\delta_j^{(1)}$ and $\delta_j^{(2)}$, $j = 1, \dots, q$, such that

$$\begin{aligned} |s(\varphi_j - \delta_j^{(1)})| &= |s(\varphi_j + \delta_j^{(2)})| := h_j, \\ |s(x)| &\leq h_j \quad \text{for } x \in [\varphi_j - \delta_j^{(1)}, \varphi_j + \delta_j^{(2)}], \end{aligned}$$

$j = 1, \dots, q$, and $U_\rho(w) \cap Z = \emptyset$, where

$$Z = \bigcup_j [\varphi_j - \delta_j^{(1)}, \varphi_j + \delta_j^{(2)}].$$

Now,

$$\begin{aligned} \|f - sr_n\|_{L_p(w)} &\leq \left\| s \left| \frac{f}{s} - r_n \right| \chi_{[0, 1] \setminus Z} \right\|_{L_p(w)} \\ &\quad + \| |f - sr_n| \chi_Z \|_{L_p(w)} := H_1 + H_2. \end{aligned}$$

Define a continuous function g on $[0, 1]$ as follows:

$$g(x) = \begin{cases} \frac{f(x)}{s(x)} & \text{for } x \in [0, 1] \setminus Z, \\ \text{linear} & \text{otherwise.} \end{cases}$$

By Lemma 3, it is easy to show that there exist $r_n \in R_n$, $n \geq n_0$, such that

$$H_1 \leq \|s\|_\infty \|g - r_n\|_{L_p(w)} \rightarrow 0, \quad (10)$$

as $n \rightarrow \infty$, $\{r_n\}$ converges to g uniformly on Z , and (7) holds. Hence, for all large n , we have

$$\sup_{x \in [\varphi_j - \delta_j^{(1)}, \varphi_j + \delta_j^{(2)}]} |r_n(x)| \leq 2 \|f\|_\infty h_j^{-1}, \quad j = 1, \dots, q.$$

It follows that

$$H_2 \leq \|f\|_\infty \|\chi_Z\|_{L_p(w)} + 2 \|f\|_\infty \sum_{j=1}^q h_j^{-1} \|s \chi_{[\varphi_j - \delta_j^{(1)}, \varphi_j + \delta_j^{(2)}]}\|_{L_p(w)}. \quad (11)$$

According to the assumption (i) of the theorem, for any given $\varepsilon > 0$, we can choose $\delta_j^{(1)} > 0$ and $\delta_j^{(2)} > 0$, $j = 1, \dots, q$, such that

$$\|\chi_Z\|_{L_p(w)} < \varepsilon.$$

Hence, we obtain

$$H_2 \leq C_q \|f\|_\infty \varepsilon \quad (12)$$

for some constant C_q depending only on q . Combining (10), (11), and (12), we arrive at

$$e_n(s, f)_{L_p(w)} \rightarrow 0,$$

as $n \rightarrow \infty$. This completes the proof of the theorem.

4. APPROXIMATION ORDER

We will establish the following result.

THEOREM 2. *Let the classes $A(\Gamma, U)$ and $A(\Delta, V)$ be given as above, $0 < p \leq \infty$, and s and w satisfy the conditions in Theorem 1. If $s(x) > 0$ for all $x \in [0, 1]$, then*

(i) there exist $A > 1$ and $B > 1$ such that for every f in $A(\Gamma, U)$

$$e_n(s, f)_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathcal{E}_n(B)),$$

(ii) there is a $\lambda > 0$ such that for every f in $A(\Gamma, U)$

$$e_n(s, f)_{L_p(w)} = O(e^{-\lambda \sqrt{n}})$$

whenever

$$\max_{\theta_j \in \Gamma \cup \Delta} \mu_j < \min_{1 \leq s \leq m, 1 \leq s' \leq l} \{u_s + 1, v_{s'} + 1\}. \quad (13)$$

Here,

$$\mathcal{E}_n(B) = \sum_{\theta_s \in \Gamma \cup \Delta} \min(\mathcal{E}_{n,s}^-(B), \mathcal{E}_{n,s}^+(B))$$

with $\mathcal{E}_{n,s}^-(B)$ and $\mathcal{E}_{n,s}^+(B)$ defined, similar to the notations used in Lemma 3, as follows:

(1) If $\theta_s = x_j \in \Gamma \setminus \Delta$, then

$$\begin{aligned} \mathcal{E}_{n,s}^-(B) &= (B^{-\sqrt{n}})^{u_j+1} \|\chi_{[x_j-\delta, x_j-B^{-\sqrt{n}}]}\|_{L_p(w)} \\ &\quad + \|\chi_{[x_j-B^{-\sqrt{n}}, x_j]}(\cdot)(\cdot - x_j)^{u_j+1}\|_{L_p(w)}, \\ \mathcal{E}_{n,s}^+(B) &= (B^{-\sqrt{n}})^{u_j+1} \|\chi_{[x_j+B^{-\sqrt{n}}, x_j+\delta]}\|_{L_p(w)} \\ &\quad + \|\chi_{[x_j, x_j+B^{-\sqrt{n}}]}(\cdot)(\cdot - x_j)^{u_j+1}\|_{L_p(w)}. \end{aligned}$$

(2) If $\theta_s = y_j \in \Delta \setminus \Gamma$, then $\mathcal{E}_{n,s}^-(B)$ and $\mathcal{E}_{n,s}^+(B)$ are defined as above with the exception that x_j and u_j are replaced by y_j and v_j , respectively.

(3) If $\theta_s = x_{j_1} = y_{j_2} \in \Delta \cap \Gamma$, then

$$\begin{aligned} \mathcal{E}_{n,s}^-(B) &= (B^{-\sqrt{n}})^{\min(u_{j_1}, v_{j_2})+1} \|\chi_{[\theta_s-\delta, \theta_s-B^{-\sqrt{n}}]}\|_{L_p(w)} \\ &\quad + \|\chi_{[\theta_s-B^{-\sqrt{n}}, \theta_s]}(\cdot)(\cdot - \theta_s)^{\min(u_{j_1}, v_{j_2})+1}\|_{L_p(w)} \end{aligned}$$

and $\mathcal{E}_{n,s}^+(B)$ is defined similarly.

We now sketch a proof of this result. If $s > 0$, then it follows that

$$\|f - s r_n\|_{L_p(w)} \leq \|s\|_\infty \|g - r_n\|_{L_p(w)}, \quad (14)$$

where $g = f/s \in A(\Gamma^*, U^*)$, $\Gamma^* = \Gamma \cup \Delta = \{z_1, \dots, z_{m'}\}$ with $z_1 < z_2 < \dots < z_{m'}$, $U^* = \{u_1^*, \dots, u_{m'}^*\}$, and $u_j^*, j = 1, \dots, m'$, defined by

$$u_j^* = \begin{cases} u_j & \text{if } z_j \in \Gamma \setminus \Delta, \\ v_j & \text{if } z_j \in \Delta \setminus \Gamma, \\ \min(u_j, v_j) & \text{if } z_j \in \Delta \cap \Gamma. \end{cases}$$

Hence, there is a polynomial p_0 of degree $\leq \sum_{j=1}^{m'} u_j^* + m'$ such that

$$\begin{aligned} g(x) - p_0(x) &= \sum_{j=0}^{m'} (x - x_j)^{u_j^*+1} (x - x_{j+1})^{u_{j+1}^*+1} g_j(x) \chi_{I_j}(x) \\ &= \sum_{j=0}^{m'} f_j(x) \end{aligned} \quad (15)$$

say, where $I_j = [z_j, z_{j+1}]$, $j=0, \dots, m'$, and g_j is analytic on I_j . By Lemma 3 and Remark 1, we see that there exist $A_j > 1$ and $B_j > 1$, $j=0, \dots, m'$, such that

$$e_n(f_j)_{L_p(w)} = O(A_j^{-\sqrt{n}}) + O(\mathcal{E}_n(B_j)), j=0, 1, \dots, m'. \quad (16)$$

Then (14), (15), and (16) together give the conclusion (i) in Theorem 2.

If the condition (13) is satisfied, then it is easy to see that

$$\mathcal{E}_n(B_j) = O(\bar{B}^{-\sqrt{n}})$$

for some $\bar{B}_j > 1$. Thus, (ii) follows from (i).

Remark 2. Condition (13) cannot be deleted. In fact if there is a $\theta_{j_0} \in \Delta \cap \Gamma$ such that $\mu_{j_0} = \min\{u_s + 1, v_s + 1\}$, then conclusion (ii) of Theorem 2 does not hold.

5. APPROXIMATION OF PIECEWISE SMOOTH FUNCTIONS

We need some notation. Suppose that Γ and U are given as in Section 1 and q is a positive integer such that $\max_j u_j < q$. Denote by $C^q(\Gamma, U)$ the collection of all complex-valued continuous functions f on $[0, 1]$ whose restrictions on each $I_j = [x_j, x_{j+1}]$ belong to $C^q(I_j)$, the class of functions with q th order continuous derivatives on I_j , and satisfy the joining conditions

$$f^{(s)}(x_j^-) = f^{(s)}(x_j^+), \quad s = 0, \dots, u_j,$$

with

$$f^{(u_j+1)}(x_j^-) \neq f^{(u_j+1)}(x_j^+)$$

for $j=0, \dots, m$.

By modifying the proofs in the above discussions, we have also established analogous results for the class $C^q(\Gamma, U)$. We state these results without proof.

THEOREM 3. Let s be a given function in $A(\Delta, V)$, $0 < p \leq \infty$, and w a given weight function on $[0, 1]$. Then a necessary and sufficient condition for $e_n(s, f)_{L_p(w)} \rightarrow 0$ as $n \rightarrow \infty$, where f is an arbitrary function in $C^q(\Gamma, U)$, is that the conditions of Theorem 1 are satisfied and $\mu_j \leq q$ for all $j = 1, \dots, k$.

THEOREM 4. Let s and w satisfy the conditions in Theorem 3 with $0 < p \leq \infty$. If $s(x) > 0$ for all $x \in [0, 1]$, then

(i) there exists $B > 1$ such that for every f in $C^q[\Gamma, U)$

$$e_n(s, f)_{L_p(w)} = O(\mathcal{E}_n(B)) + O\left(\frac{1}{n} \sum_{j=0}^m \omega\left(f_j, \frac{1}{n}\right)_{L_p}\right).$$

where f_j denotes the restriction of f on I_j and $\omega(f_j, 1/n)_{L_p}$ the L_p -modulus of continuity of f_j , and

(ii)

$$e_n(s, f)_{L_p(w)} = O\left(\frac{1}{n^q} \sum_{j=0}^m \omega\left(f_j, \frac{1}{n}\right)_{L_p}\right)$$

provided

$$\max_{\theta_j \in \Gamma \cup \Delta} \mu_j < \min_{1 \leq s \leq m, 1 \leq s' \leq l} \{u_s + 1, v_{s'} + 1\}.$$

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